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A THEORETICAL FRAMEWORK FOR STUDENTS' CONCEPTUAL UNDERSTANDING IN THE EARLY CALCULUS CLASSROOM

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In this article, a classification model for constructing tasks requiring students' conceptual understanding in the field of functions and the early calculus classroom is proposed. The "3L-model" combines three relevant dimensions: the involved representations (in the model called layouts) of a given function, the involved mental images (called layers), and the levels of the appearing functional relationships. The latter comprises the level of simple functions, the level of transformed functions, and the level of differentiated functions. The use of the 3L-model in test construction is illustrated for a sample item. The administration of the developed test allows insights of the conceptual understanding of more than 3000 students regarding functions in the early calculus classroom.

INTRODUCTION

In order to measure and develop a deeper understanding of students' conceptual knowledge in the field of functional relationships and differential calculus, an achievement test, the so-called FALKE-test[1], was elaborated. To ensure that the task design results into items which essentially focus on the conceptual understanding of students, a classification model – the so-called 3L-model – was developed. In the following, we will outline the theoretical background of this framework and show how tasks can be classified using it. The test focuses on students' mental images of differentiation as well as the meaning of the parameters of a function, e.g. the addition of a constant. These contents are to be taught compulsorily by the end of the first year of upper secondary mathematics (grade ten, approximately age 16) in the German federal state of North Rhine-Westphalia. This is why, we implemented the FALKE-test as an assessment of students' understanding at the end of the schoolyear in grade ten classrooms.

SETTING A THEORETICAL FRAMEWORK

When introducing students to the early calculus classroom, an adequate conceptual understanding of the associated notions and concepts is a key component of a successful mastery. Hiebert and Carpenter (1992) define understanding as follows: "A mathematical idea of procedure or fact is understood if it is part of an internal network. More specifically, the mathematics is understood if its mental representation is part of a network of representations. The degree of understanding is determined by the number and the strength of the connections" (p. 67). To ensure constructing tasks for understanding, we therefore set three main foci:

First, when working with functions some kind of representation is needed, e.g. a term, a graph or at least a verbal description of the functional relationship's characteristics. Following Hiebert and

[1] FALKE is the German acronym for: **F**unktionales Denken und **A**nalysis – **L**ernen von **K**onzepten in der **E**inführungsphase (functional thinking and analysis – learning of concepts in early calculus), see http://www.falke-test.de/

Carpenter (1992), these representations should not only exist as external visualizations, but also as parts of the students' "internal networks". It is known that the ability to use different representations and to translate between them in a flexible way is a manifestation of this internalization. Thus, asking students to work with and translate between different functional representations assesses their mathematical competence (Duval, 2006). However, it is also an activity essentially needed when learning mathematics (ibid). In order to solve tasks concerning the field of functions, students need to identify the function's characteristics in the given representation of it. In this sense, a learner might discover different properties of a specific function, if he or she is expected to work with a graph rather than a term. In the following, we will call this dimension of a test item, namely the type of representation given in a task, the *layouts of a function*.

Second, the learner's own mental images play a crucial role when teaching and learning the concept of function as they constitute central aspects of a learner's internal network. In the German tradition of subject-matter didactics, these images are commonly referred to as "Grundvorstellungen" (*GV*) (vom Hofe & Blum, 2016). Focusing on the concept of function, there is a consensus in the German literature that three main GVs exists: The function as a mapping of values, the function as the covariation of two variables, and the function as an independent mathematical object with its own properties and characteristics (cf. Ruchniewicz & Barzel, 2019; vom Hofe, 2001). From an international perspective, e. g. Dubinksy and Harel (1992) describe broadly similar aspects of the notion of function, namely the action, process, and object concept. We will refer to these mental images as the *layers of a function*.

Third, one has to recognize that functions can exist on different levels (Hahn & Prediger, 2008). In its simplest form, a task given to a student comprises only stand-alone functions. In differential calculus, commonly two different but connected functions play a role: a function and its derivative. Therefore, students have to alternate/shift between two levels of the same function while regarding that changes on one level influence the other. Hahn and Prediger (2008) call this "Funktionsebene" which we translate as the *levels of a function*. Similarly, when working with transformations of functions, such as $g(x) \coloneqq 2f(x)$, this can be seen as two different but interacting levels (Klinger, 2018).

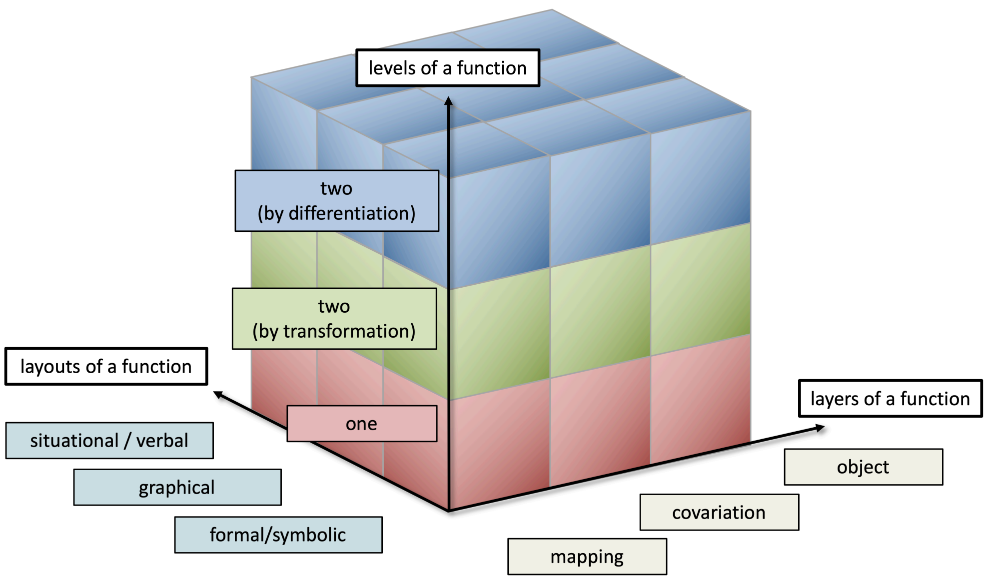

Figure 1: 3L-model used for item and test design within the FALKE test (Klinger, 2018).

USING THE 3L-MODEL FOR TASK-DESIGN

Grasping these distinctions as three separate dimensions for classifying tasks, leads to the *3L-model for task-design* (displayed in Figure 1). Every task touches a certain number of the model's sub-cubes. The underlying assumption is, that the more sub-cubes are associated with an item, the more parts of the students' internal network have to be activated and therefore, the more it focuses on assessing conceptual understanding.

It should be mentioned that we do not distinguish between the directions in which any of the sub-cubes interact: e.g. there is no distinction if a student is asked to construct a symbolic term from a function's graph or vice versa. On the one hand, the direction in which a representation is translated is in practice hard to identify (Janvier 1978, p. 3.3f). On the other hand, this reduces the models' complexity. Moreover, the model does not include tables of values as a form of representation since these were not included in the constructed test. However, it is possible to extend the classification model accordingly or to add a third type of "level two functions", when the inclusion of integrals is needed.

The following example will show how the 3L-model can be used for task design: In the sample item in figure 2 students have to imagine the transformation of a function as well as corresponding interactions with its derivative. They need to recognize that only horizontal movements survive differentiation, so that (d) is the correct solution. The item integrates two possible *layouts*, a graphical perspective (since it displays the given functions as graphs) as well as a verbal description ("f is moved by…"). It is of primary importance to grasp the occurring functions as independent mathematical objects and to operate between them, to master this item successfully. Therefore, "object" is the main *layer* here. Finally, it includes operations on all *levels* of functions: initial function (green cubes), level two by transformation (red cubes) and level two by differentiation (blue cubes). The item therefore touches the sub-cubes as displayed on the right-hand side of figure 2.

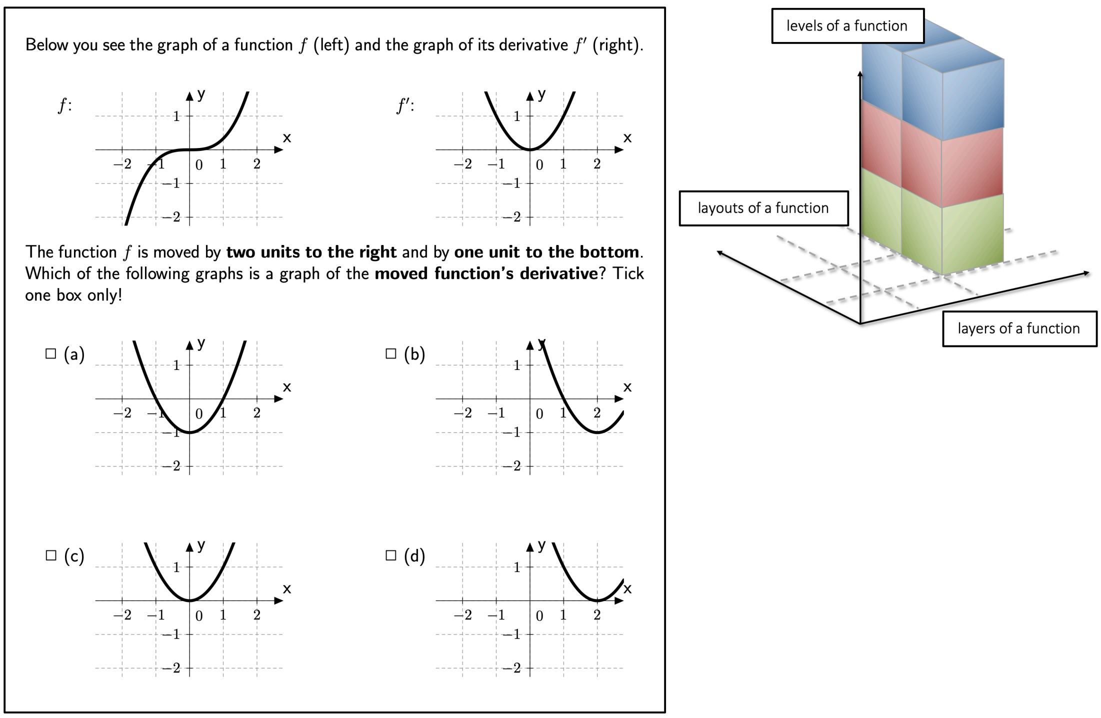

Figure 2: Sample item (left) and its associated sub-cubes of the 3L-model (right) (item translated from Klinger 2018, p. 309).

RESULTS FROM TESTING THE TASKS IN THE FIELD

Our sample consists of students from the first year of upper secondary level (approx. age 16) in Germany. We compiled the constructed items into two different tests and administrated them at the very beginning and the very end of the schoolyear, testing 3202 and 2665 students, respectively. The first test consists of 18, the latter of 26 items.

The considered sample item was solved by 35.7 percent of the students. Answer (b) is the strongest distractor, uniting 54.0 percent of all attempts. With just 4.0 and 5.5 percent possibility (a) and (c) appear as comparatively weak distractors. The majority of students chooses a graph in which the manipulations to the initial function f were directly transferred to its pretended derivative. Since other items (not shown here) allow the conclusion that far more students understand the concept of transformation as well as differentiation in an isolated manner, one might infer that most students are incapable of bringing both concepts together. Thus, according to the 3L-model, not grasping the connection between functions of the 2nd level by transformation and the 2nd level by differentiation. This is just one example where students are not able to connect two concepts to the desired extent.

In conclusion, our study suggests that a) the 3L-model can be a useful tool when designing tasks assessing conceptual understanding of functions in the calculus classroom and b) setting a stronger focus on connecting different topics within the calculus classroom would be beneficial for many of the learners. Future work could investigate the 3L-model as a potential tool for the design of learning tasks instead of test items.

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